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TECHNICAL NOTES

Non-linear convection in a porous medium with inclined temperature gradient and vertical throughflow

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1. INTRODUCTION

The problem of thermal convection in thin porous horizontal layers, either uniformly heated from below (the Bénard problem) or differentially heated at the side walls (Hadley circulation), are now well understood. In physical situations, strict uniform heating either only in the vertical or only in the horizontal direction rarely occurs. Generally, both the horizontal and vertical temperature gradients are present simultaneously. Weber [1] was the first author to consider general linear instability of a convection problem in a porous medium, induced by an inclined temperature gradient by means of a perturbation method. Nield [2] removed the restriction in [1], and used a low-order Galerkin approximation to solve the associated eigenvalue problem. Later on Nield [3] noted that his earlier treatment [2] was not satisfactory, and employed a higher-order Galerkin approximation and found considerably improved results. Kaloni and Qiao [4] have discussed the non-linear stability of the title problem in the absence of vertical throughflow via the energy method. These authors have used the compound matrix method to solve the associated eigenvalue problem and the golden-section search method to carry out the maximum and minimum routines. The authors have also carried out the linear stability calculations and their findings compare reasonably well with Nield's [3] result. For the non-linear problem, the results of Kaloni and Qiao [4], predicted the possibility of subcritical instability.

Our purpose here to extend the results of Kaloni and Qiao [4] when vertical throughflow is present. Apart from being a new scientific investigation this problem has relevance to the performance of packed bed reactors. We again employ the energy method for a non-linear stability analysis and carry out both linear and non-linear stability calculations. For linear instability, we find that our results are in good agreement with those of Nield's [5] for low values of the horizontal Rayleigh numbers. We, however, give complete results for higher values of horizontal Rayleigh number. In addition, we also provide new non-linear stability results and these predict the possibility of subcritical instabilities.

2. BASIC EQUATIONS

We consider a porous medium occupying a layer of height H. The z-axis is chosen vertically upwards and x-axis is in the direction of applied horizontal temperature gradient β_T. The vertical temperature difference across the boundaries is ΔT. We assume that the flow in the porous medium is governed by Darcy's law. For the density variation, the Boussinesq approximation is assumed to be valid. Accordingly, following the non-dimensionalization scheme of Nield [3], the governing equations then take the form.

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\mathbf{v} + \nabla P = T\mathbf{k} \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \nabla^2 T \tag{3}$$

where $\mathbf{v}$, P, and T are the non-dimensionalized seepage velocity, pressure and temperature, respectively, and $\mathbf{k}$ is the unit vector in the z-direction. We assume there is a throughflow of velocity w_v in the vertical direction. The non-dimensional boundary conditions thus take the form

$$w = Q_v, \quad T = -(\pm R_V/2) - R_H x \quad \text{at } z = \pm 1/2 \tag{4}$$

where $Q_v = w_v H/\alpha_m$ is the Péclet number, R_V and R_H are vertical and horizontal Rayleigh numbers, respectively, and are defined as

$$R_V = \frac{\rho_0 g \gamma_T K H \Delta T}{\mu \alpha_m}, \quad R_H = \frac{\rho_0 g \gamma_T K H^2 \beta_T}{\mu \alpha_m}. \tag{5}$$

Here ρ_0 is the density at the reference temperature, g is the gravitational acceleration, γ_T is the thermal expansion coefficient, K is the permeability of the medium, μ is the dynamic viscosity, and α_m is the thermal diffusivity.

The basic steady state solution $(\mathbf{u}_s, T_s, p_s)$ of equations (1)–(3) satisfying the boundary condition (4) is

$$u_s = R_H z, \quad v_s = 0, \quad w_s = Q_v \tag{6}$$

$$T_s = -R_H x + f(z) \tag{7}$$

$$\nabla p_s = T_s \mathbf{k} - \mathbf{u}_s \tag{8}$$

where

$$f(z) = \frac{R_H^2}{8Q_v}(4z^2-1) + \frac{R_H^2 z}{Q_v^2} - \frac{(R_H^2 + Q_v^2 R_V)}{2Q_v^2 \sinh(Q_v/2)}[\exp(Q_v z) - \cosh(Q_v/2)]. \tag{9}$$

We remark that we have imposed the requirement that there is no horizontal mass flow, namely,

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NOMENCLATURE

a	dimensionless overall horizontal wave number
D	differential operator, d/dz
g	gravitational acceleration
H	layer height
i, j, k	unit vectors in the x, y and z-directions, respectively
K	permeability
l	dimensionless wave number in y-direction
p_s	dimensionless steady state pressure
P, p	dimensionless pressure/perturbed pressure
Q_v	Péclet number
R_E	vertical energy Rayleigh number
R_H	horizontal Rayleigh number
R_L	vertical linear Rayleigh number
t	dimensionless time
T	dimensionless temperature
T_s	dimensionless steady state temperature
$\mathbf{u}$, $\mathbf{v}$	dimensionless perturbed velocity vector/velocity vector
$\mathbf{u}_s$	dimensionless steady state velocity vector
w_v	dimensionless velocity of vertical throughflow
x, y, z	dimensionless Cartesian coordinates.

Greek symbols

α_m	thermal diffusivity
β_T	horizontal temperature gradient
γ_T	coefficient of volume expansion
θ	perturbed dimensionless temperature
κ	thermal diffusivity
μ	dynamic viscosity
ρ_0	density at the reference temperature.

$$\int_{-1/2}^{1/2} u_s\,dz = 0, \quad \int_{-1/2}^{1/2} v_s\,dz = 0 \tag{10}$$

3. STABILITY ANALYSIS

We now perturb the basic-state solution as follows:

$$\mathbf{v} = \mathbf{u}_s + \mathbf{u}, \quad T = T_s + \theta, \quad P = p_s + p. \tag{11}$$

The perturbation equations then take the form

$$\nabla \cdot \mathbf{u} = 0 \tag{12}$$

$$\mathbf{u} + \nabla p = \theta \mathbf{k} \tag{13}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \nabla^2 \theta - \mathbf{u}_s \cdot \nabla \theta - \mathbf{u} \cdot \nabla T_s \tag{14}$$

where $\mathbf{u}_s$ and T_s are given by equations (6)–(9). The corresponding boundary conditions then become

$$w = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{15}$$

We introduce a positive coupling parameter ξ and define an energy functional $E(t)$, as

$$E(t) = \frac{\xi}{2}\|\theta\|^2. \tag{16}$$

On multiplying equation (13) by $\mathbf{u}$ and equation (14) by θ and integrating over a typical periodic cell denoted by V, we can derive (cf. ref. [4]),

$$\frac{dE}{dt} = I - M \tag{17}$$

where

$$I = -\xi\langle(\mathbf{u} \cdot \nabla T_s)\theta\rangle + \langle \theta w \rangle \tag{18}$$

$$M = \xi\|\nabla\theta\|^2 + \|\mathbf{u}\|^2. \tag{19}$$

Here $\langle\cdot\rangle$ denotes the integration over V, and $\|\cdot\|$ denotes the $L^2(V)$ norm. We now define

$$m = \max_{\mathscr{H}} \frac{I}{M} \tag{20}$$

where $\mathscr{H}$ is the space of admissible solutions. On combining equation (17) with equations (18)–(20), and by using Poincaré inequality, we can infer, for $0 < m < 1$, that

$$\frac{dE}{dt} \leqslant -2\pi^2(1-m)E. \tag{21}$$

Inequality (21) clearly indicates that for $0 < m < 1$, $E(t) \to 0$ at least exponentially as $t \to \infty$.

We now consider the maximum problem given by equation (20) at the critical argument $m = 1$. The associated Euler–Lagrange equations become

$$-\xi \nabla T_s \cdot \mathbf{u} + w + 2\xi \nabla^2 \theta = 0 \tag{22}$$

$$\xi \nabla T_s \theta - \theta \mathbf{k} + 2\mathbf{u} = \nabla \omega \tag{23}$$

where ω is a Lagrange multiplier introduced since $\mathbf{u}$ is solenoidal.

On taking curlcurl of equation (23) and then considering the third component of the resulting equation we find

$$2\nabla^2 w - \nabla_1^2(g_1\theta) + \xi R_H \frac{\partial^2 \theta}{\partial x \partial z} = 0 \tag{24}$$

where $\nabla_1^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ and

$$g_1 = (1 - \xi f_1),$$

$$f_1 = \frac{R_H^2}{Q_v^2}(Q_v z + 1) - \frac{(R_H^2 + Q_v^2 R_V)}{2Q_v \sinh(Q_v/2)}\exp(Q_v z). \tag{25}$$

The other equations to be used along with equation (24) are

$$\xi R_H u + g_1 w + 2\xi \nabla^2 \theta = 0 \tag{26}$$

$$-\xi R_H \theta + 2u = \frac{\partial \omega}{\partial x}. \tag{27}$$

Following Nield [3] and Kaloni and Qiao [4], we restrict our attention to the steady longitudinal mode analysis as it is the most favorable mode of disturbance in the absence of vertical throughflow. Accordingly, we look for a solution, of the above equation, in the form

$$(u, w, \theta, \omega) = [u(z), w(z), \theta(z), \omega(z)]\exp(ily). \tag{28}$$

On eliminating variables u and ω, we derive the eigenvalue problem, which can be written as

$$D^2 w = h_1 w + h_2 \theta \tag{29}$$

$$D^2 \theta = h_3 w + h_4 \theta \tag{30}$$

where $D = \mathrm{d}/\mathrm{d}z$, $a^2 = l^2$, and $h_1, \ldots, h_4$ are given as:

$$h_1 = a^2, \quad h_2 = -\frac{a^2 g_1}{2}, \quad h_3 = -\frac{g_1}{2\xi},$$

$$h_4 = a^2 - \frac{\xi R_H^2}{4}. \tag{31}$$

The relevant boundary conditions are

$$w = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{32}$$

4. NUMERICAL RESULTS

We now consider R_V as the eigenvalue with the remaining variables: ξ, R_H, a^2, Q_v, as parameters. The critical vertical Rayleigh number is defined by

$$R_E = \max_{\xi} \min_{a^2} R_V(R_H, a^2, \xi, Q_v). \tag{33}$$

On letting $x_1 = w$, $x_2 = Dw$, $x_3 = \theta$, $x_4 = D\theta$, the system of equations (29) and (30) can be written in the matrix form as

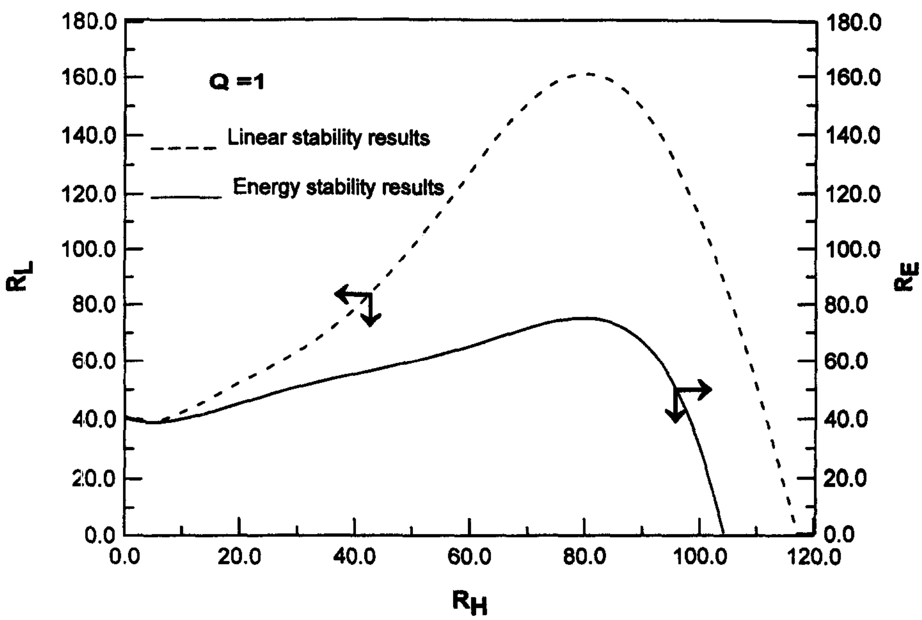

Fig. 1. Critical vertical linear R_L and energy R_E Rayleigh numbers vs. horizontal Rayleigh numbers for Péclet number $Q_v = 1$.

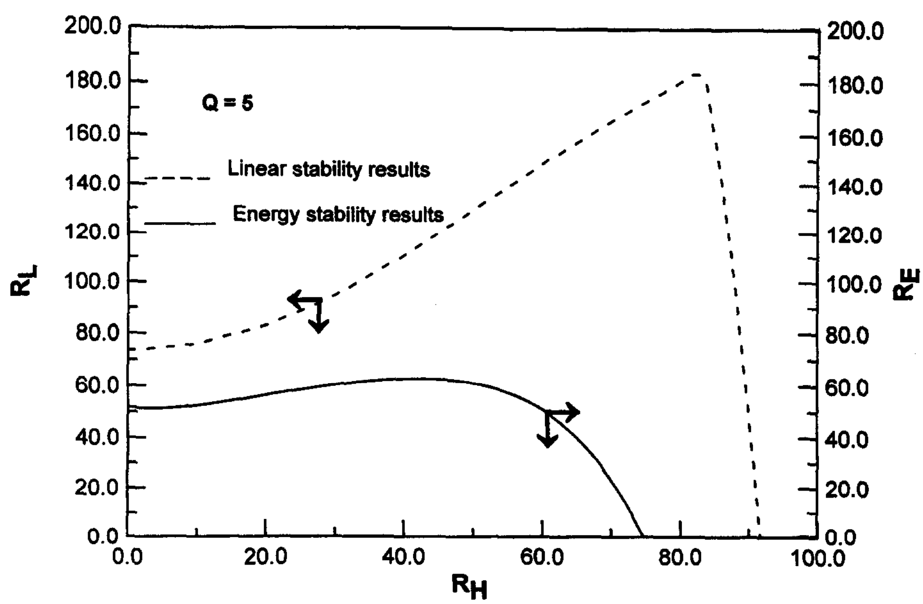

Fig. 2. Critical vertical linear R_L and energy R_E Rayleigh numbers vs. horizontal Rayleigh numbers for Péclet number $Q_v = 5$.

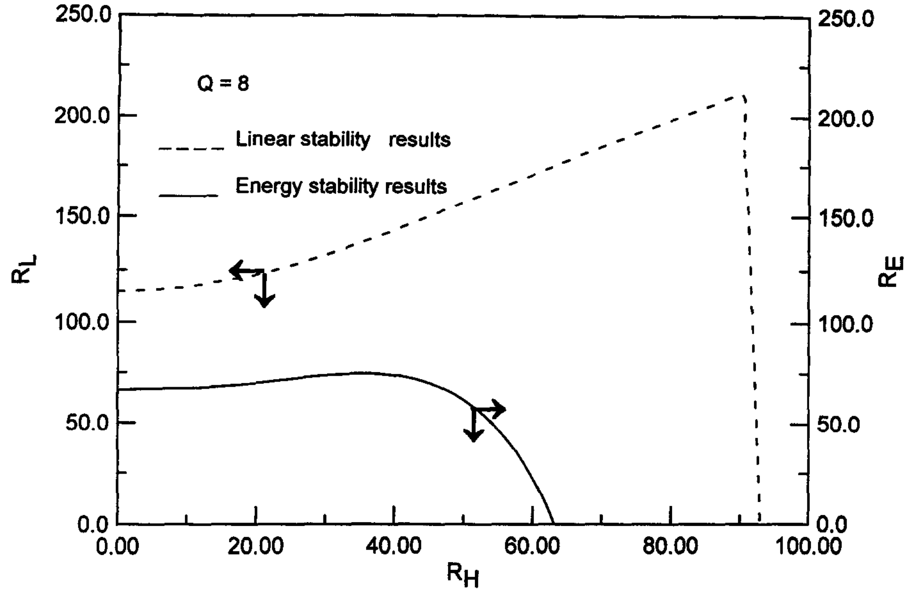

Fig. 3. Critical vertical linear R_L and energy R_E Rayleigh numbers vs. horizontal Rayleigh numbers for Péclet number $Q_v = 8$.

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \tag{34}$$

where $\mathbf{X} = (x_1, x_2, x_3, x_4)^{\perp}$ and $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ h_1 & 0 & h_2 & 0 \\ 0 & 0 & 0 & 1 \\ h_3 & 0 & h_4 & 0 \end{bmatrix}. \tag{35}$$

The boundary conditions now take the form

$$x_1 = x_3 = 0 \quad \text{at } z = \pm 1/2. \tag{36}$$

We next employ the compound matrix method and carry out the maximization and minimization routines by the golden-section search. Figures 1–3 display our computed results of linear and non-linear critical vertical Rayleigh numbers for different values of the Péclet number Q_v.

In the figures that follow we have plotted the critical vertical Rayleigh numbers, for both linear and energy methods, for various values of R_H and Q_v. These are denoted by R_L and R_E, respectively. Our aim here is to look for the effect of increasing R_H and Q_v on these critical parameters. First of all we remark that our linear instability results nearly coincide with those of Nield [5] for low values of R_H. Thus, in this range, the effect of increasing R_H and Q_v values is almost additive and has the effect of stabilizing both R_L and R_E values. A further increase in R_H values, however, have the destabilizing effect and we find that destabilization starts earlier for lower values of Q_v. We may conclude that higher values of Q_v have the stabilizing effect. Thus, even though the mechanism of throughflow delays the onset of convection it cannot completely control the instability caused by the higher values of horizontal applied gradient. We remark that results in [5], which are reported for the values of R_H up to 40, do not reflect this destabilizing behaviour. Indeed the destabilizing in the linear case only starts after R_H has taken values higher than 70. The energy stability results are almost parallel to the linear instability results except that the changes from stabilizing to destabilizing occur at the lower values of R_H. This fact is understandable because energy method results are usually more conservative. In this case again increasing Q_v seems to have stabilizing effect.

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REFERENCES

1. Weber, J. E., Convection in a porous medium with horizontal and vertical temperature gradients. *International Journal of Heat and Mass Transfer*, 1974 **17**(2), 241–248.
2. Nield, D. A., Convection in a porous medium with inclined temperature gradient. *International Journal of Heat and Mass Transfer*, 1991, **34**(1), 87–92.
3. Nield, D. A., Convection in a porous medium with inclined temperature gradient: additional results. *International Journal of Heat and Mass Transfer*, 1994, **37**(18), 3021–3025.
4. Kaloni, P. N. and Qiao, Z.C., Non-linear stability of convection in a porous medium with inclined temperature gradient. *International Journal of Heat and Mass Transfer*, 1997, **40**(7), 1611–1615.
5. Nield, D.A., Convection in a porous medium with inclined temperature gradient and vertical throughflow. *International Journal of Heat and Mass Transfer*, 1998, **41**(1), 241–243.